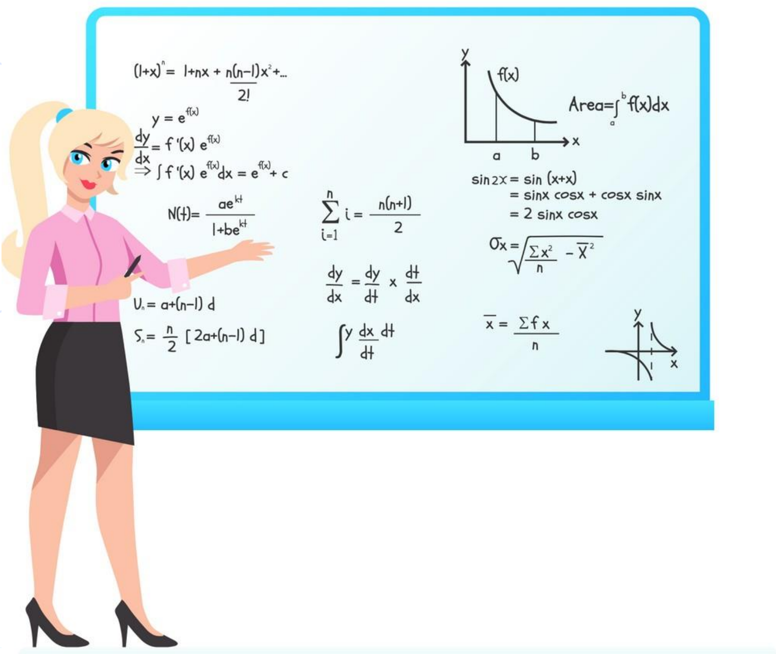
Real Analysis

Textbook and Course Advice



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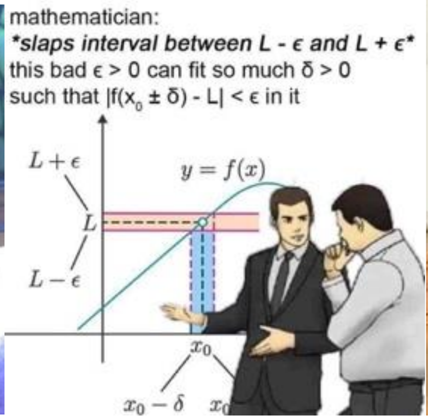
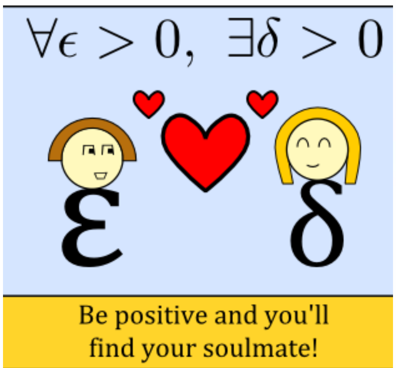
Real Analysis:

Table of Contents





1 General Advice

This is usually the dreaded course of year 1. Do not try and avoid Analysis. It will make you a better mathematician!

Don't panic if you're struggling a lot with this course at first, relax and give it time. Read lots, do lots of practise exercises and all the homework set. Use your teacher's recommended textbook(s) plus some of the ones mentioned in this document. Analysis is a massive branch of Mathematics, and your teacher will only select some things to teach you. Know exactly what topics are in your course, so you don't need to learn too much extra information at first. If your teacher's preferred text is unclear, then switch to another text to learn the topic, but switch back when you can so that you are aware of the resources that your teacher expects you to have.

Analysis is the formality of Calculus – it is "why calculus works" and if you think of it from that angle, it at least blesses you with a little bit of context, which I think is invaluable. "Calculus" by Spivak is a great book for this.

Understand sequences well. If your course is anything like a typical Analysis course, everything will build off of sequences, series and functions. You will require an excellent grasp of sequences and functions will set the stage for derivatives and integrals. In addition, revisit set theory. It's the foundation of Mathematical Analysis. Knowing BASIC Topology is important. Begin with a book called "Introduction to Topology" by Bert Mendelson to get a grasp on the big picture. Then read "Topology" by Munkres if you have time. Topology is also a huge area, so it's quite easy to get lost at the beginning!

2 Fun Background Information

Analysis is the attempt to apply rigour to our understanding of mathematics. A lot of concepts you meet in your early academic life, and when studying mathematics in other disciplines, make a lot of "intuitive" assumptions. Analysis aims to put rigour into these methods.

In 1920, David Hilbert started an initiative to provide a solid foundation for all of mathematics based on a system of axioms. A noble ambition. A few years later Kurt Gödel proved that this was impossible to do! Poor Gödel – he became mentally unstable. He only trusted his wife's cooking. When she became ill, he refused to eat and starved to death. Nevertheless, most of mathematics, thanks to David Hilbert, Bernhard Riemann, Augustin-Louis Cauchy, George Cantor and others, is now on a solid foundation.

It is "intuitive" that, to be differentiable, a function must be continuous. However, you will meet functions which are continuous everywhere but not differentiable anywhere!

What is "continuous" anyway? You will meet a formal method for defining this. Known as the epsilon-delta method (devised by Cauchy) this applies the necessary rigour. Many students have difficulty understanding this, so a full explanation and examples are given in the appendix.

We've all met infinite series. Examples are:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Do these have finite values, or do they diverge? The results may surprise you. There are many methods you will meet to help you decide.

But what is infinity? Did you know there are two types of infinity? David Hilbert designed a hotel with infinity number of rooms. It was always full. But there was always room for more guests – an infinite number of them! You will meet his hotel as you study analysis.

Clearly you can assign an integer to every counting number.

$$1, 2, 3, 4, \cdots$$

Surprisingly that can do the same for all rational numbers! George Cantor showed have this can be done. But you can't do the same for irrational numbers. Again, Cantor showed that this is true. He then spent the rest of his life looking for some other type of infinity but ended his life in a mental sanitorium.

You will discover all of this during your studies.

Newton & Leibnitz devised calculus. But this was only put onto a formal basis by Bernhard Riemann many centuries later. You will meet this during your studies. Riemann integration is fine for continuous functions. But what about functions that are discontinuous everywhere? Well, you can still integrate these. Henri Lebesgue showed how this can be done.

3 Textbooks

3.1 Beginner

- **Understanding Analysis, Abbott**
  This is a clear book and THE BEST place to start (I recommend finishing "How to Prove" it by Velleman or any other proof technique before starting this book though). "How to solve it" by Polya or "Thinking Mathematically" by Mason are also great!
  This textbook gives the motivation for studying analysis and is the favourite of most students. Just bear in mind that Abbot treats topological concepts in $\mathbb{R}$ in terms of sequences at first.
  The proofs and examples are explained in great detail and there are plenty of exercises
  Check out Michael Penn's YouTube channel. He has a full, free excellent course based on this textbook.

- **Elementary Analysis: The Theory of Calculus, Kenneth Ross**
  This is a great book for a beginner since it treats analysis in a very detailed way. It even has discussion of proofs before the formal proofs. This books also deals with point set topology in $\mathbb{R}^n$. You can find exercise solutions online.

- **Introduction to Real Analysis, Bartle**
  This is also a good book (except the integration part, where Bartle took a relatively ad hoc approach). It can be helpful with certain proofs. The style of explanation is much more student friendly than Rudin.
  Bartle also has a book named "Elements of Real Analysis." It is a sequel to this book and deals with analysis in $\mathbb{R}^n$. It is a very good book with lots of figures to explain abstract concepts as connected sets. However, this book does not cover metric spaces (it is only covered in the exercises section).

- **Mathematical Analysis, Binmore**

- **How to think about Analysis, Alcock**

3.2 Advanced

- **Calculus, Spivak**

- **Analysis 1, Tao**

- **A First Course in Real Analysis, Protter**

- **Real Analysis, Royden**

- **Real Mathematical Analysis, Pugh**

- **Calculus Vol I and II, Apostol**

- **Examples and Theorems in Analysis, Walker**
  Lots of students love this book, however It is not for everyone.

- **Principles of Mathematical Analysis, Rudin**
  This is definitely not a year 1 book, rather more a year 2.

- **A Basic Course in Real Analysis, Kumaresan**
  This has the same amount of rigor as Rudin, but a much better way of presenting the proofs.

3.3 Comparison and Evaluation of Advanced Textbooks

- Spivak versus Apostol:
  While both books have complete proofs and a good emphasis on theory, Spivak's book is better as an introduction to rigorous math because many of its problems are more difficult and theoretically oriented than Apostol's. Spivak's book also has a solution manual, which is very useful when you're studying on your own.

On the other hand, Apostol actually covers more material, even just within Volume 1, and also has a greater variety of exercises involving applications of calculus to physics. Volume 2 of Apostol is actually one of the best introductions to multivariable calculus. If you have time, learning from both Spivak and Apostol is a good idea. Otherwise, if your main focus is pure maths, then I would recommend Spivak.

- Rudin versus others
  Rudin is very hard to read and rather difficult for a first course. The point of a first course is to teach you how to work with the standard analytical methods like epsilons and sequences, but this book already assumes that you have good experience with analysis and takes it a step too far when it comes to avoiding spoon feeding. Rudin defers a lot of the work to the reader, so you will find yourself often having to scribble on a piece of draft paper in order to walk yourself through what's been written. He does very little to truly explain and many proofs in the book are more sketches than real proofs. There are some horrible parts of the book like the one where he proves the Rank Theorem and the last chapter about Measure Theory is just hideous. That said, there are some truly remarkable sections (mainly from 1 to 7).

  If you want to read this book, I suggest reading Stephen Abbott's “Understanding Analysis” first and then Apostol's book after if time allows - it is a lot friendlier than Rudin's. You will then be in a good position to go through Rudin’s after having had exposure to Abbott and Apostol.

  So, in summary, this book is very good book for revision, but certainly not suitable for beginners unless you are a genius or very hardworking!

3.4 Rudin’s Three Textbooks

The bulk of “Principles of Mathematical Analysis” (Baby Rudin) is devoted to a rigorous introduction to single variable and multivariable calculus. The last few chapters definitely do not bridge the gap between baby Rudin and papa Rudin (Real and Complex Analysis). There are 3 editions of this book, each one is different from its predecessor.

In contrast, “Real and Complex Analysis” (Papa/Big Rudin) covers measure theory, some functional analysis, Fourier analysis and complex analysis. It is a graduate level analysis and combines what are usually graduate level real and complex analysis classes that PhD students take in their first or second year. This book is excellent, concise and well formulated.

“Functional Analysis” (Grandpa Rudin) is an excellent graduate level book.

In my estimation, all three classics mentioned above are gems and outstanding if you want to really get into the heart of the maths evolving in the nineteenth and early twentieth centuries. The elegance and beauty of Rudin's expositions is impressive!

4 Videos

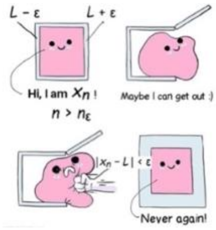
- Professor Francis Su
https://www.youtube.com/playlist?list=PL0E754696F72137EC&fbclid=IwAR3f6YVlTVM96fDENdPXgU3NpTkgNT_piIPwcyD9imsF9e4yBqGRg_n2Kk4

- Michael Penn
There is a full free excellent course on Real Analysis. This is based on Abbot's book as mentioned earlier. He explains everything in a detailed and concise way and the proofs start with a "sketch proof" which is so helpful.
https://www.youtube.com/playlist?list=PL22w63XsKjqxqaF-Q7MSyeSG1W1_xaQoS

5 Appendix

Familiarise yourself with the following epsilon delta definitions below before starting your course. The notation can be daunting for students when first starting an Analysis course.

5.1 ε definition of convergence (for a sequence)



A sequence $\{x_n\}$ converges to a real number $l \in \mathbb{R}$ if the following holds
$\forall\ \varepsilon > 0, \exists N > 0\ s.t.\ n > N \Rightarrow |x_n - l| < \varepsilon$

We say l is the limit of the sequence $\{x_n\}$ and we commonly write this as:
$\lim_{n\to\infty} x_n = l$ or $\lim x_n = l$ or $x_n \longrightarrow l$ as $n \longrightarrow \infty$

What does this formula intuitively mean? Let's break each part down:

- ✓ $\forall\ \varepsilon > 0$:
  The number epsilon gets introduced as a positive number here, but it only gets some real meaning at the end
- ✓ $\exists N > 0\ s.t.\ n > N$:
  Here we just restrict the domain of x_n to $n > N$ i.e. at some point N every number in the sequence will be past the point N. At values less than N, there are a FINITE number of terms, but past that point N, an infinite number of terms in the sequence exist,
- ✓ $|x_n - l| < \varepsilon$:
  This means $l - \varepsilon < x_n < l + \varepsilon$ i.e. x_n is only a small distance ε away from the limit l which the sequence tends so within a small 'radius' (distance) of the target.
  So, in summary, at values less than N, there are a FINITE number of terms, but past that point N, an infinite number of terms in the sequence exist, all of which are within a certain tiny distance, epsilon, to the convergent value l

Show that $\lim_{n\to\infty} \frac{n+1}{2n+5} = \frac{1}{2}$

$\forall\ \varepsilon > 0, \exists N > 0\ s.t.\ n > N \Rightarrow \left|\frac{n+1}{2n+5} - l\right| < \varepsilon$

$\left|\frac{n+1}{2n+5} - \frac{1}{2}\right| = \left|\frac{2n+2-2n-5}{2(2n+5)}\right| = \left|\frac{-3}{2(2n+5)}\right| = \frac{3}{2(2n+5)}$

$so\ \frac{3}{2(2n+5)} < \varepsilon$

$\Leftrightarrow 3 < 2\varepsilon(2n+5)$

$\Leftrightarrow 4n\varepsilon + 10\varepsilon > 3$

$\Leftrightarrow n > \frac{-10\varepsilon+3}{4\varepsilon}$

So, we take N=$\frac{-10\varepsilon+3}{4\varepsilon}$

Note: you are always given the value of the limit l in the questions

5.2 ε, δ definition of a limit

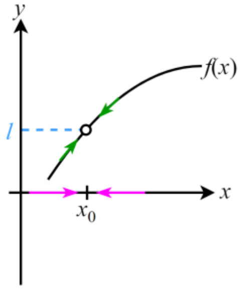
5.2.1 Notation

What does the notation $\lim_{x\to x_0} f(x) = l$ mean?

$\lim_{x\to x_0} f(x) = l$ means that as x approaches x_0, the function $f(x)$ approaches l.

Informally we say, a limit exists at a point x_0 i.e. $\lim_{x \to x_0} f(x) = l$ if we can trace THE CURVE inwards from either side of this point and tend towards the same y value of l

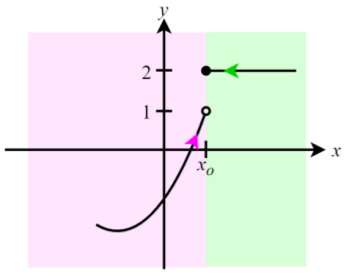
This looks like



$\lim_{x \to x_0} f(x) = l$

hence if $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$ then $\lim_{x \to x_0} f(x)$ exists. We call this limit l and write $\lim_{x \to x_0} f(x) = l$
Note: $\lim_{x \to x_0^+} f(x)$ *and* $\lim_{x \to x_0^-} f(x)$ are called one-sided limits. A one-sided limit is the value the function approaches as the x-values approach the limit from *one side only*

Consider the following example.



Let's look at this as approaching from the pink side (left limit) and the green side (right limit)

$$\lim_{x \to x_0^-} f(x) = 1 \text{ and } \lim_{x \to x_0^+} f(x) = 2$$

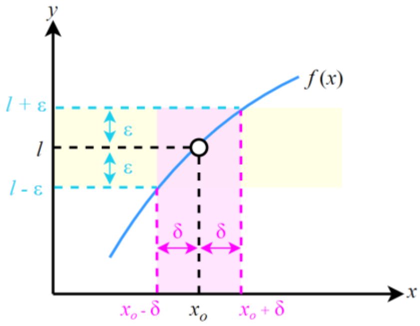
Here $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$ and hence the limit does not exist

5.2.2 Formal Definition

Formally we define the limit as,
$\lim_{x \to x_0} f(x) = l$ if given any $\varepsilon > 0, \exists \delta > 0 \;\; s.t \; \forall x \;\; 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$

This formula looks scary at first, but if you break it down it is not at all!

Visually, this looks like



It is obvious from the graph that it is saying, if you pick a value close to x_0 (within a small δ range of x_0) then you are guaranteed that $f(x)$ will not be any further than a given small distance (ε) from l.
The formula can also be written for each left and right sided limit as:

Right sided: $\lim_{x \to x_0^+} f(x) = l$ if $\varepsilon > 0, \exists \delta > 0 \; s.t. \forall \, x, \; \boldsymbol{x_0 < x < x_0 + \delta} \Rightarrow |f(x) - l| < \varepsilon$

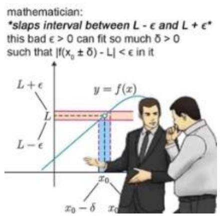
Left sided: $\lim_{x \to x_0^-} f(x) = l$ if $\varepsilon > 0, \exists \delta > 0 \; s.t. \forall \, x, \; \boldsymbol{x_0 - \delta < x < x_0} \Rightarrow |f(x) - l| < \varepsilon$

What does this formula intuitively mean? Let's break each part down

- ✓ $\varepsilon > 0$ means our proof must work for every epsilon, we have no control over epsilon
- ✓ $\exists \delta > 0$ implies that the proof will have to give the value of delta, so that the existence of that number is confirmed. Typically, the value of delta will depend on the value of epsilon. This is why you will often see δ written as $\delta(epsilon)$.
- ✓ $s.t \; \forall x$ implies we cannot restrict the values of x any further than the next restriction provides
- ✓ $0 < |x - x_0| < \delta$ which is the same as writing $x_0 - \delta < x < x_0 + \delta$.
  This is the starting point for a series of implications (algebra steps) which will conclude the final statement.
  The expression $|x - x_0| < \delta$ means the values of x will be close to x_0, specifically not more than (or even equal to) delta units away.
  The expression $0 < |x - x_0|$ implies that x is not equal to c itself
  Note:
  For the left hand limit we write $x_0 - \delta < x < x_0$
  For the right hand limit we write $x_0 < x < x_0 + \delta$
- ✓ $|f(x) - l| < \varepsilon$ is the same as writing $l - \varepsilon < f(x) < l + \varepsilon$
  This is the conclusion of the series of implications. Once this statement is reached, the proof will be complete

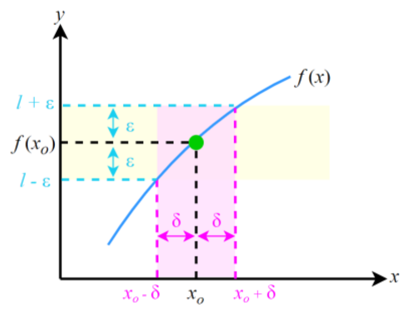
Upon examination of these steps, we see that the key to the proof is the identification of the value of delta. To find that delta, we typically begin with the final statement $|f(x) - l| < \varepsilon$ and work backwards until we reach the form $|x - x_0| < \delta$.

5.3 ε, δ definition of continuity



A function $f(x)$ is continuous at the point x_0 if
$\forall \, \varepsilon > 0, \exists \delta > 0 \; s.t. \forall \, x, \; 0 \leq |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Visually, this looks like,



$\lim_{x \to x_0} f(x) = f(x_0) = l$ and $f(x_0)$ is defined (notice how the point at x_0 is filled in now in green)

This is the same as the above limit definition except for continuity we need to have an equality on zero (hence the filled in point in green). This is because not only do we need the limit to exist, but we ALSO want $f(x_0)$ to be defined and equal to the value of the limit for $f(x)$ to be continuous.
Some courses don't write that there must be an equality to zero

Above, we saw that
$\forall \, \varepsilon > 0, \exists \delta > 0 \; s.t. \forall \, x, \; 0 \leq |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Instead, we can say
$\forall\, \varepsilon > 0, \exists \delta > 0\ s.t. \forall\, x,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$
This is like saying $0 \le x < 2$ is the same as $x < 2$, as long as we know (and it's obvious) that x can't take negative values

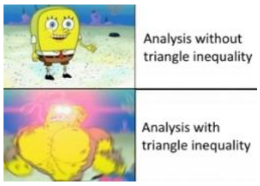
$f(x) = \begin{cases} 3x + 2 & if\ x \le 2 \\ x^3 & if\ x > 2 \end{cases}$.
Prove using ε and δ that $f(x)$ is continuous at $x = 2$
So, we need to prove that $\lim_{x\to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 8$ and that $f(2) = 8$
Let's first prove $\lim_{x\to 2} f(x) = 8$ using ε and δ def^n of limit

- $\boldsymbol{x < 2}$:
  Let $\varepsilon > 0$ be given.
  We need to find a $\delta > 0\ s.t\ \mathbf{2 - \delta < x < 2} \Rightarrow |f(x) - 8| < \varepsilon$
  $f(x) = 3x + 2$
  $|f(x) - 8| < \varepsilon$
  $\Leftrightarrow |3x + 2 - 8| < \varepsilon$
  $\Leftrightarrow |3x - 6| < \varepsilon$
  $\Leftrightarrow -\varepsilon < 3x - 6 < \varepsilon$
  $\Leftrightarrow 6 - \varepsilon < 3x < 6 + \varepsilon$
  $\Leftrightarrow \frac{6-\varepsilon}{3} < x < \frac{6+\varepsilon}{3}$
  $\Leftrightarrow 2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}$
  Compare with $2 - \delta < x < 2$
  $2 - \frac{\varepsilon}{3} = 2 - \delta \Rightarrow \delta = \frac{\varepsilon}{3}$
  So take $\delta = \frac{\varepsilon}{3}$
  Check $\delta > 0$
  This is true since $\varepsilon > 0$
- $\boldsymbol{x > 2}$:
  Let $\varepsilon > 0$
  We need to find a $\delta > 0\ s.t\ \mathbf{2 < x < 2 + \delta} \Rightarrow |f(x) - 8| < \varepsilon$
  $f(x) = x^3$
  $|f(x) - 8| < \varepsilon$
  $\Leftrightarrow |x^3 - 8| < \varepsilon$
  $\Leftrightarrow -\varepsilon < x^3 - 8 < \varepsilon$
  $\Leftrightarrow 8 - \varepsilon < x^3 < 8 + \varepsilon$
  $\Leftrightarrow \sqrt[3]{8 - \varepsilon} < x < \sqrt[3]{8 + \varepsilon}$
  Compare with $2 < x < 2 + \delta$
  $\sqrt[3]{8 + \varepsilon} = 2 + \delta \Rightarrow \delta = \sqrt[3]{8 + \varepsilon} - 2$
  So take $\delta = \sqrt[3]{8 + \varepsilon} - 2$
  Check $\delta > 0$
  This is true since $\varepsilon > 0$
  So $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 8$
  Now, since $f(2) = 8$, the function is continuous at $x = 2$

Learn all the epsilon delta proofs in your course. Do not memorise them, understand them!

5.4 Triangle Inequality

The triangle inequality will be used over and over again! If you don't know this you won't pass analysis!



$|a + b| \leq |a| + |b|$

Similarly, $\big||a| - |b|\big| \leq |a + b|$

So together we get, $\big||a| - |b|\big| \leq |a + b| \leq |a| + |b|$

You'll see this can be written in many forms:

- $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$
  Why?
  $|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$

- $|a + b| \geq |a| - |b|$

- $|a - b| \geq |a| - |b|$

Putting all together we have

$|a| - |b| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ or $|a| - |b| \leq |a - b| \leq |a| + |b|$

And similarly,

$|b| - |a| \leq |a + b| \leq |b| + |a|$ or $|b| - |a| \leq |a - b| \leq |b| + |a|$